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Robust output regulation of a class of uncertain nonlinear minimum phase systems perturbed by unknown external disturbances ☆

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ABSTRACT

This paper addresses the problem of robust output regulation for uncertain nonlinear minimum phase systems affected by an unknown external disturbance. We present a novel unknown input observer (UIO)-embedded high-gain stabilizer to completely reconstruct the disturbance in a finite time. A new saturation function is introduced to prevent finite time escape of the closed-loop trajectories. This allows us to achieve a semi-globally exponential regulation of the regulated output. The proposed method features the exact cancellation rather than approximate cancellation, for both structured and/or unstructured external disturbance. The presented controller is finite-dimensional and does not require the nominal model of the plant. Finally, a numerical experiment is conducted to demonstrate the effectiveness of the proposed method.

1. Introduction

The robust output regulation problem, namely having the output of a system asymptotically tracking prescribed trajectories and rejecting unwanted disturbances simultaneously, in the presence of parameter uncertainties, has received everlasting attention in the control community. Especially in complex practical problems, for instance, the attitude control of a quadrotor (Yang, Cheng, Xia, & Yuan, 2017), active noise control (Airimitoaie, Landau, Melendez, & Dugard, 2020; Kamaldar & Hoagg, 2018) and flexible joint robot control (Psomopoulou, Theodorakopoulos, Doulgeri, & Rovithakis, 2015), etc., strong nonlinearities and various uncertainties can be found ubiquitously. More importantly, disturbances in the process industry are highly likely to be unmeasurable and unstructured. Hence, regardless of the great progress (Bernard, Bin, & Marconi, 2020; Ran, Wang, & Dong, 2016; Serrani, Isidori, & Marconi, 2001), providing alternative solutions with high performance and capability of rapidly eliminating disturbances remains imperative in both theoretical and engineering fields of control.

Existing techniques for disturbance rejection can be roughly categorized into robust control approaches and internal model (IM)-based methods. The former, including disturbance observer (DOB) control (Sariyildiz, Oboe, & Ohnishi, 2019), active disturbance rejection control (ADRC) (Han, 2009), extended high-gain observer-based control (Freidovich & Khalil, 2008; Wang, Isidori, & Su, 2015), just name a few, are capable of handling both structured and unstructured disturbances. In DOB, a key step is to design a Q-filter whose bandwidth is crucial to robust stability, but is usually selected by trial and error. A constructive method for the determination of the Q-filter's bandwidth has been recently developed in Chang, Kim, and Shim (2020), but the systems considered in Chang et al. (2020), Shim and Jo (2009) are restricted to be linear and minimum-phase. For nonlinear systems, several nonlinear versions of DOB have been developed (Back & Shim, 2008; Ding, Chen, Mei, & Murray-Smith, 2019; Ha & Back, 2023). Nevertheless, such a criterion of selecting Q-filter's bandwidth is inappropriate in the nonlinear content. Besides, accurate information about the nonlinear function of the plant model is demanded in Ding et al. (2019). Alternatively, the ADRC technique becomes favorable in practice due to its simplicity for implementation, however, the rigorous theoretical proof of the stability analysis is nontrivial (Chi, Hui, Huang, & Hou, 2021; Jiang, Huang, & Guo, 2015; Ran et al., 2016). Moreover, one main drawback shared by all the aforementioned robust control approaches is that the disturbance rejection is approximate (Back & Shim, 2008; Chi et al., 2021; Ha & Back, 2023), which limits their application in tasks requiring high precision.

Rather than approximate cancellation, the *Internal Model Principle* provides a necessary condition for asymptotic cancellation, saying a suitable model of the disturbance must be reduplicated in the closed-loop systems (Francis & Wonham, 1976). This IM-based method is feasible when the model or the parameter of the exogenous model is

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exactly known. Such an assumption, however, is rather impractical in many real-world scenarios. To overcome this difficulty, an adaptive identifier designed by Lyapunov-based method (Serrani et al., 2001) or optimization method (Bernard et al., 2020; Bin, Marconi, & Teel, 2019) for the internal model suffices to reduplicate the disturbance model. However, the cancellation result is directly related to the performance of the identifier. The nonzero identification error results in a residual error between the disturbance model and internal model. Then, still only approximate cancellation is achieved. Moreover, if unstructured disturbances are considered, it is claimed in Bin, Astolfi, and Marconi (2022) that for nonlinear systems, no finite-dimensional internal model exists for asymptotic regulation. Regarding the disturbance as an unknown input, the UIO technique, from a geometric perspective, provides an alternative solution to exact cancellation, but it requires complete knowledge of the systems (Alenezi, Zhang, Hui, & Żak, 2021; Corless & Tu, 1998; Yang, Barboni, Rezaee, Serrani, & Parisini, 2022).

Inspired by the above discussions, a natural question arises: *for nonlinear uncertain systems perturbed by possibly unstructured disturbances, does there exist a finite-dimensional regulator to achieve asymptotic cancellation of unstructured disturbances and meanwhile guarantee robust stability?* In this work, this question is positively answered for a class of uncertain nonlinear SISO systems with arbitrary relative degrees. We consider the systems to be minimum-phase, and notably, the nonlinear functions are completely unknown and state-dependent, which is more general than the nonlinearity output-dependent or with approximated expressions (bounding functions, constant bounds) (Cruz-Zavala, Moreno, & Nuño, 2022; Mercado-Uribe & Moreno, 2020; Sanchez & Moreno, 2021).

Our approach aligns with the DOB technique and ADRC methods (Chi et al., 2021; Ran et al., 2016; Sariyildiz et al., 2019), where unstructured disturbances and uncertain dynamics are amalgamated as a lumped perturbation to be compensated. However, our subsequent steps differ significantly. To precisely estimate this lumped uncertainty, we initially transform the robust output regulation problem into an unknown input observation problem through a meticulously crafted coordinate change. Subsequently, we propose a novel UIO-based high-gain control protocol. Specifically, we introduce a high-order sliding mode (HOSM) differentiator to estimate the unmeasurable state, and employ a novel interval observer-based UIO as proposed in Zhu, Fu, and Dinh (2023) to achieve an exact estimation of the lumped uncertainty. Yet, compensating by directly substituting the estimated lumped uncertainty may lead to a finite-time escape of closed-loop trajectories, causing instability. To mitigate this issue, we introduce a new saturation function in the control law. In this way, the attraction region of the augmented zero dynamics can be arbitrarily increased, which makes it possible to utilize a high-gain feedback combined with its estimate to preserve the closed-loop trajectories within a given forward invariant set. By maintaining states within this set, all estimates exhibit finite-time convergence due to the introduction of a series of HOSM differentiators. As a result, we achieve complete cancellation of unstructured disturbances while ensuring demonstrable closed-loop stability.

The main contributions of this paper are summarized as follows:

(i) A novel high-gain feedback law combined with an interval observer-based UIO is presented to achieve semi-global asymptotic stability for largely uncertain nonlinear minimum phase systems perturbed by unmeasurable external disturbances. The proposed controller requires no prior information of the disturbance structure, and more importantly, achieves the precise and direct cancellation. The key secret lies in the use of the proposed interval observer-based UIO, instead of IM-based solutions (Bernard et al., 2020; Bin et al., 2019).

(ii) The proposed controller, in which a saturation function is introduced upon the estimation term and along with the high gain feedback, is able to alleviate the transient overshoot of the closed-loop trajectories such that the trajectories remain in a given forward invariant set. Consequently, the proposed control protocol ensures the validity of the nonlinear separation principle (Khalil & Praly, 2014), allowing for the independence of stability analyses for both estimators and system states.

(iii) The proposed method can easily tackle the nonlinear system with completely unknown nonlinear functions, since the nonlinear separation principle holds. As a result, the proposed method is applicable to a broader class of systems, in contrast to approaches in Cruz-Zavala et al. (2022), Mercado-Uribe and Moreno (2020), Sanchez and Moreno (2021), which rely on an approximated expression and homogeneous Lyapunov function for closed-loop systems.

The remainder of the paper is organized as follows. Section 2 gives problem formulation and some standing assumptions. In Section 3, a high-gain-based control law is proposed, together with a novel interval-observer-based UIO. The stability analysis is presented in Section 4. Numerical examples are provided to illustrate the performance of the proposed controller compared with other disturbance rejection methods Bernard et al. (2020) and Back and Shim (2008) in Section 4. The paper is wrapped up with conclusions in Section 6.

Notations: $\|\cdot\|$ represents the Euclidean norm of the matrices or vectors; For any constant matrix or vector $M \in \mathbb{R}^{m\times n}$ ($\mathbb{R}^m$), $M > (\geq, <, \leq)\ 0$ means that all elements of M are $> (\geq, <, \leq)\ 0$ respectively. Denote $M^+ = \max\{M, 0\}$ and $M^- = \max\{-M, 0\}$. Then obviously, we have $M = M^+ - M^-$ and $|M| = M^+ + M^-$, where $|M|$ stands for a $m\times n$ matrix ($m \times 1$ vector) formed by taking the absolute value of every element of M. In addition, a Metzler matrix is a square matrix whose off-diagonal components are all non-negative. Define the function $\lceil\alpha\rfloor^\gamma = |\alpha|^\gamma \operatorname{sign}(\alpha)$, for any $\gamma \in [0, 1)$ and any $\alpha \in \mathbb{R}$. In this paper, the solution of discontinuous differential equations is understood in Filippov's definition (Filippov, 2013).

2. System description and problem formulation

2.1. System description

Consider a class of SISO nonlinear system with a well-defined relative degree r and rewritten in the Byrnes–Isidori normal form (Isidori, 2017; Khalil, 1996):

$$\begin{aligned}
\dot{z} &= f_0(z, \xi_1, \mu),\\
\dot{\xi}_i &= \xi_{i+1}, \quad i = 1, \dots, r-1\\
\dot{\xi}_r &= q(z, \xi, \mu) + b(z, \xi, \mu)(u + d),\\
y &= \xi_1
\end{aligned} \tag{1}$$

with state variables $z \in \mathbb{R}^n$, $\xi := (\xi_1, \dots, \xi_r) \in \mathbb{R}^r$, control input $u \in \mathbb{R}$, *unmeasurable* disturbance $d \in \mathbb{R}$ and regulated output $y \in \mathbb{R}$. The uncertain parameter vector μ is supposed to range over a given compact set $\mathcal{P} \subset \mathbb{R}^p$. The *unknown* nonlinear functions $f_0(\cdot), q(\cdot), b(\cdot)$ are sufficiently smooth functions, with $b(\cdot)$ representing the high-frequency gain of system (1). Without loss of generality, $f_0(0, 0, \mu) = 0$ and $q(0, 0, \mu) = 0$ such that the point $(z, \xi) = (0, 0)$ is the equilibrium. Let $\mathrm{Z}_0 \subset \mathbb{R}^n$ and $\mathrm{X}_0 \subset \mathbb{R}^r$ be any fixed compact sets. For (1), the initial conditions $z_0(\mu) := z(0)$ and $\xi_0(\mu) := \xi(0)$ depend on μ, and fulfill $(z_0(\mu), \xi_0(\mu)) \in \mathrm{Z}_0 \times \mathrm{X}_0$ for all $\mu \in \mathcal{P}$.

Systems in the form of (1) are representative in the literature focusing on the output regulation problem, which fit in many practical applications (Psomopoulou et al., 2015; Yang et al., 2017). As commonly seen in the topic of nonlinear output regulation (Back & Shim, 2008; Bernard et al., 2020; Bin et al., 2019; Isidori, Marconi, & Praly, 2012; Serrani et al., 2001), we assume the system is minimum phase, and the sign and the bound of $b(z, \xi, \mu)$ are known. Without loss of generality, $b(z, \xi, \mu)$ is assumed to be uniformly positive. The suggested solution is readily adaptable to scenarios involving a negative sign. To be specific:

Assumption 2.1. The zero dynamics

$$\dot{z} = f_0(z, 0, \mu) \tag{2}$$

is locally asymptotically stable with an attraction region $\mathcal{A}$ satisfying $\mathrm{Z}_0 \subset \mathcal{A} \subset \mathbb{R}^n$.

Assumption 2.2. The high-frequency gain $b(z, \xi, \mu)$ is assumed to be bounded away from zero and there exist two known positive constants $\underline{b}, \overline{b}$ such that

$$0 < \underline{b} \leq b(z, \xi, \mu) \leq \overline{b}, \quad \forall (z, \xi, \mu) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathcal{P}. \tag{3}$$

The disturbance, denoted as $d(t)$ in this context, can exhibit either structured or unstructured characteristics. It is worth noting that this study places a particular emphasis on evaluating the effectiveness of the proposed controller in handling *unstructured* external signals, as it does not rely on any prior knowledge about the model of disturbances. However, the bounds of disturbance and the initial condition of system (1) are required to be known as prior, as stated in the following assumption:

Assumption 2.3. There exist two known constant vectors $\overline{\xi}(0)$ and $\underline{\xi}(0)$ confirming to $\underline{\xi}(0) \leq \xi_0(\mu) \leq \overline{\xi}(0)$ for all $\mu \in \mathcal{P}$. Moreover, there exist known positive constants d_1 and d_2 for the uniformly bounded disturbance d such that for all $t \geq 0$, $|d(t)| \leq d_1$ and $|\dot{d}(t)| \leq d_2$.

Although Assumptions 2.2–2.3 indicate that all uncertainties are bounded and the bounds are known, the bounds are non-conservative in the sense that we can always select a sufficiently large boundary for a bound term. However, given a known compact set for initial conditions, it is regrettable to acknowledge that the regulation result is semi-global.

Remark 2.1. Note that, the nonlinear functions in (1) are unknown, in the sense that we do not require the precise or approximated expression (bounding functions, constant bounds) of the nonlinear functions (Cruz-Zavala et al., 2022; Mercado-Uribe & Moreno, 2020; Sanchez & Moreno, 2021), thus enlarging the class of considered systems. $\triangleleft$

In this setting, a rough description of the robust output regulation problem is described as: for an *uncertain* system (1) perturbed by some *unmeasurable* disturbance $d(t)$, find a controller with the only measurable output $y(t)$ such that the trajectories of the closed-loop system originating from $\mathrm{Z}_0 \times \mathrm{X}_0$ are bounded and the regulated output satisfies $\lim_{t\to\infty} y(t) = 0$.

2.2. Problem formulation

Consistently with most of the literature (Byrnes & Isidori, 1991; Isidori et al., 2012; Serrani et al., 2001) treating the case $r > 1$, in this work we augment the zero dynamics (2) with additional zeros to seek semi-global regulation. To be specific, we introduce the following coordinate change:

$$\begin{aligned} &\xi_i \mapsto e_i := k^{-(i-1)}\xi_i, \quad i = 1, \ldots, r-1 \\ &\xi_r \mapsto e_r := \xi_r + k^{r-1}\alpha_0\xi_1 + k^{r-2}\alpha_1\xi_2 + \cdots + k\alpha_{r-2}\xi_{r-1} \end{aligned} \tag{4}$$

in which $k > 1$ is a sufficiently large constant and parameters α_i, $i = 0, \ldots, r-2$ are selected such that the characteristic polynominal $s^{r-1} + \alpha_{r-2}s^{r-2} + \cdots + \alpha_0$ is Hurwitz. With the coordinate change (4) and denote $\mathbf{e} := (e_1, \ldots, e_{r-1}) \in \mathbb{R}^{r-1}$ for simplicity, system (1) becomes

$$\begin{aligned} \dot{z} &= f_0(z, e_1, \mu), \\ \dot{\mathbf{e}} &= kA\mathbf{e} + Be_r, \\ \dot{e}_r &= b(z, \xi, \mu)u + \delta(z, \mathbf{e}, e_r, d, \mu) \end{aligned} \tag{5}$$

in which

$$A = \begin{pmatrix} \mathbf{0}_{(r-2)\times 1} & \mathbf{I}_{r-2} \\ -\alpha_0 & \bar{\alpha}_{1\times(r-2)} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbf{0}_{(r-2)\times 1} \\ 1/k^{r-2} \end{pmatrix}$$

with $\{\bar{\alpha}\}_{1,i} = -\alpha_i$, $i = 1, \ldots, r-2$ and the nonlinear term $\delta(z, \mathbf{e}, e_r, u, d, \mu)$ is given by

$$\begin{aligned} \delta(z, \mathbf{e}, e_r, d, \mu) :=\, & b(z, \xi, \mu)d + q(z, \xi, \mu) \\ & + k^{r-1}\alpha_0\xi_2 + k^{r-2}\alpha_1\xi_3 + \cdots + k\alpha_{r-2}\xi_r. \end{aligned} \tag{6}$$

However, system (5) is still not suitable for the controller design as $b(z, \xi, \mu)$ is unknown. Hence, we add and subtract a term βu to the last equation in (5), and rewrite (5) into

$$\begin{aligned} \dot{z} &= f_0(z, \mathbf{C}\mathbf{e}_a, \mu), \\ \dot{\mathbf{e}}_a &= \mathbf{A}\mathbf{e}_a + \mathbf{B}\beta u + \mathbf{B}\Delta(z, \mathbf{e}_a, u, d, \mu), \\ y &= \mathbf{C}\mathbf{e}_a \end{aligned} \tag{7}$$

in which $\mathbf{e}_a := (\mathbf{e}, e_r) \in \mathbb{R}^r$, β is a positive constant, $\Delta(z, \mathbf{e}_a, u, d, \mu)$ is the lumped uncertainty described by

$$\Delta(z, \mathbf{e}_a, u, d, \mu) = \delta(z, \mathbf{e}, e_r, d, \mu) + (b(z, \xi, \mu) - \beta)u \tag{8}$$

with $\delta(z, \mathbf{e}, e_r, d, \mu)$ defined in (6), and

$$\mathbf{A} = \begin{pmatrix} kA & B \\ \mathbf{0}_{1\times(r-1)} & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0_{(r-1)\times 1} \\ 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & \mathbf{0}_{1\times r-1} \end{pmatrix}. \tag{9}$$

Note that, when $r = 1$, we do not require to take the coordinate change as (4) in the case $r > 1$. However, for the sake of compactness and neatness, we denote $\mathbf{e}_a = e_r = \xi_1$ when $r = 1$. Then, the form of (7) holds for $r \geq 1$.

When all states of (7) are measurable, it is intuitive to design an ideal control law as

$$u^* = -\frac{le_r + \Delta(z, \mathbf{e}_a, u, d, \mu)}{\beta} \tag{10}$$

for (7), in which the second term of (10) is to compensate for the lumped uncertainty, and the first term of (10) is designed such that the following closed-loop system

$$\begin{aligned} \dot{z} &= f_0(z, e_1), \\ \dot{\mathbf{e}} &= kA\mathbf{e} + Be_r, \\ \dot{e}_r &= -le_r. \end{aligned} \tag{11}$$

is semi-globally asymptotically stable (Byrnes & Isidori, 1991, Theorem 7.2). Such a controller (10), however, is unfortunately impossible for practical implementation since e_r and $\Delta(z, \mathbf{e}_a, d, \mu)$ are unmeasurable. Hence, we recast the robust output regulation problem into an observer design problem as follows:

Problem 2.1. Suppose Assumptions 2.1–2.3 hold. Let $\overline{\mathrm{Z}}_0 \in \mathbb{R}^{n+r-1}$ and $\overline{\mathrm{X}}_0 \in \mathbb{R}$ be compact sets such that $(z(0), \mathbf{e}(0)) \in \overline{\mathrm{Z}}_0$ and $e_r(0) \in \overline{\mathrm{X}}_0$. For (7), develop an observer-based regulator with $\hat{e}_r$ and $\hat{\Delta}$ in the form of

$$\hat{e}_r = \phi_1(y), \quad \hat{\Delta} = \phi_2(y)$$

for some functions $\phi_1(\cdot), \phi_2(\cdot)$ such that the following properties hold:

(i) the estimated error $\tilde{e}_r := \hat{e}_r - e_r$ and $\tilde{\Delta} := \hat{\Delta} - \Delta$ are bounded and $\lim_{t\to\infty} \tilde{e}_r(t) = 0$, $\lim_{t\to\infty} \tilde{\Delta}(t) = 0$;
(ii) the closed-loop trajectories of (7) are bounded and the regulated output satisfies $\lim_{t\to\infty} y(t) = 0$.

In what follows, for the sake of clarity, we omit the augment μ when no confusion is caused.

3. Controller design

Following the structure of (10), we propose a certainty equivalent control law u in the form of

$$u = -\frac{l\hat{e}_r + \mathrm{Sat}[\hat{\Delta}]}{\beta} \tag{12}$$

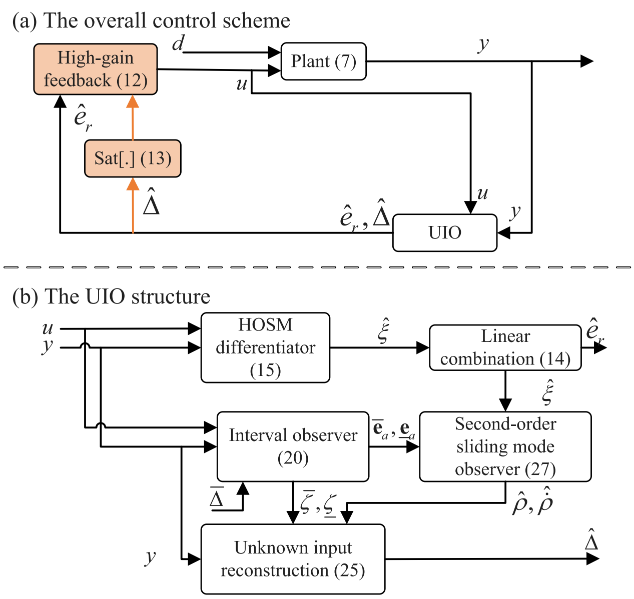

Fig. 1. (a) Schematic of the UIO-embedded high-gain control protocol, where estimates $\hat{e}_r$ and $\hat{\Delta}$ are described in Sections 3.1 and 3.2 respectively. (b) The structure of the proposed interval observer-based UIO.

in which β defined in (7) and l are positive parameters selected according to (35) and (40), $\hat{e}_r, \hat{\Delta} \in \mathbb{R}$ will be given by a high-order sliding model (HOSM) differentiator in Section 3.1 and a novel interval observer-based UIO in Section 3.2 respectively. Moreover, Sat$[\cdot]$ is a smooth "saturation function"[1] in the form of

$$\mathrm{Sat}\,[s] = \begin{cases} s, & |s| \le \vartheta \\ s - \mathrm{sign}(s)\frac{(|s|-\vartheta)^2}{2}, & \vartheta < |s| < \vartheta + 1 \\ (\vartheta + \frac{1}{2})\,\mathrm{sign}(s), & |s| \ge \vartheta + 1 \end{cases} \tag{13}$$

with a saturation parameter ϑ given by (38). Fig. 1 pictorially shows a glance at the control protocol.

Here, along with the feedback law (12), the saturation function Sat$[\cdot]$ is introduced upon the estimation term $\hat{\Delta}$ to alleviate the transient overshoot of the closed-loop trajectories such that the trajectories remain in a given forward invariant set, which in turn ensures the existence of the solutions. Nevertheless, such a conclusion requires rigorous proof (see the proof in Theorem 4.1). For now, we cannot exclude a prior the finite time escape of the closed-loop trajectories (7) under the control input (12), unless the *nonlinear separation principle* (Khalil & Praly, 2014) is suitable for stabilization using the HOSM differentiator[2]. Temporarily, we assume there exists a forward invariant set for system (7), which ensures the stability analysis of the estimators and system states can be independent. To be specific, with the initial conditions in (7) set within the compact set $\overline{\mathrm{Z}}_0 \times \overline{\mathrm{X}}_0$, we assume there exist compact sets $\overline{\mathrm{Z}}$ and $\overline{\mathrm{X}}$ satisfying $\overline{\mathrm{Z}}_0 \subset \overline{\mathrm{Z}}$ and $\overline{\mathrm{X}}_0 \subset \overline{\mathrm{X}}$ such that $\mathcal{D} := \overline{\mathrm{Z}} \times \overline{\mathrm{X}}$ is a forward invariant set for system (1), i.e., the states $(z, \mathbf{e}_a) \in \mathcal{D}$ for all $t \ge 0$. Under this temporary assumption, the main purpose of the subsequent subsections is to give a systematic presentation of designing the estimates $\hat{e}_r$ and $\hat{\Delta}$ and then this assumption will be removed and such a forward invariant set will be given in Section 4.

[1] The function Sat$[s]$ is odd and monotonically increasing, with its derivative satisfying $0 < \mathrm{Sat}'[s] \le 1$.

[2] In Cruz-Zavala et al. (2022), Mercado-Uribe and Moreno (2020), Sanchez and Moreno (2021), the nonlinear functions are bounded by some known functional expressions, while in our case, the nonlinear functions are completely unknown. Thus, the *nonlinear separation principle* is the key proof concept to our stabilization result.

3.1. Estimate of e_r

Recalling the definition of e_r in (4), the estimate of e_r is given by

$$\hat{e}_r = \hat{\xi}_r + k^{r-1}\alpha_0\hat{\xi}_1 + k^{r-2}\alpha_1\hat{\xi}_2 + \cdots + k\alpha_{r-2}\hat{\xi}_{r-1} \tag{14}$$

where we resort to a HOSM differentiator to obtain the estimates $\hat{\xi}_1, \ldots, \hat{\xi}_r$, $i = 1, \ldots, r$. Denote $\hat{\xi} := (\hat{\xi}_1, \ldots, \hat{\xi}_r) \in \mathbb{R}^r$, the estimator $\hat{\xi}$ is given by

$$\begin{aligned} \dot{\hat{\xi}}_i &= \hat{\xi}_{i+1} - k_i L^{\frac{i}{r}} \lceil \hat{\xi}_1 - y \rfloor^{\frac{r-i}{r}}, \quad i = 1, \ldots, r-1 \\ \dot{\hat{\xi}}_r &= \beta u - k_r L \lceil \hat{\xi}_1 - y \rfloor^0 \end{aligned} \tag{15}$$

in which k_i, $i = 1, \ldots, r$ are a series of positive parameters and $L > 0$ is selected to be sufficiently large. It is explicit that $\hat{e}_r$ is able to converge to e_r in a finite time, if $\tilde{\xi} := \hat{\xi} - \xi$ converges to zero in a finite time, whose convergence property will be established in the following lemma.

Lemma 3.1. *Suppose that $(z, \mathbf{e}_a) \in \mathcal{D}$ for all $t \ge 0$ and Assumption 2.3 holds. Then, for the HOSM differentiator (15), there exist a positive constant L^* and a time instant $T_1 > 0$ such that for all $L \ge L^*$ and parameters k_i, $i = 1, \ldots, r$ chosen properly, the estimated error $\tilde{\xi}$ converges to zero in a finite time T_1.*

Proof. Subtracting the dynamics of ξ in (1) from (15), it follows that

$$\begin{aligned} \dot{\tilde{\xi}}_1 &= \tilde{\xi}_{i+1} - k_i L^{\frac{i}{r}} \lceil \tilde{\xi}_1 \rfloor^{\frac{r-i}{r}}, \quad i = 1, \ldots, r-1 \\ \dot{\tilde{\xi}}_r &\in -k_r L \lceil \tilde{\xi}_1 \rfloor^0 + [-L^*, L^*] \end{aligned} \tag{16}$$

with $L^* := \bar{b}d_1 + \max_{(z,\mathbf{e}_a)\in D} |q(z, \xi)| + (\beta + \bar{b})|u|$ and $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_r) \in \mathbb{R}^r$. Note that, the dynamics (16) has the same structure as Levant's HOSM differentiator (Levant, 2003). According to Levant (2003), if the parameter gain L is selected as $L \ge L^*$, the gains k_i, $i = 1, \ldots, r$, for instance $r = 4$, can be chosen as $k_4 = 1.1, k_3 = 1.5, k_2 = 2$, and $k_1 = 4$. Then, we employ the Lyapunov candidate function

$$V_{\tilde{\xi}} = \sum_{j=1}^{r-1} \beta_j Z_j(\tilde{\xi}_j, \tilde{\xi}_{j+1}) + \beta_n \frac{1}{p} |\tilde{\xi}_r|^p \tag{17}$$

proposed in Cruz-Zavala and Moreno (2019), in which $\beta_i > 0$, $i = 1, \ldots, r$, $p > 1$, $Z_j(\tilde{\xi}_j, \tilde{\xi}_{j+1}) = \frac{r_j}{p}|\tilde{\xi}_j|^{\frac{p}{r_j}} - \tilde{\xi}_j \lceil \tilde{\xi}_{j+1} \rfloor^{\frac{p-r_j}{r_{j+1}}} + (\frac{p-r_j}{p})|\tilde{\xi}_{j+1}|^{\frac{p}{r_{j+1}}}$, $j = 1, \ldots, r-1$ with $r_i = r + 1 - i$, $i = 1, \ldots, r$. According to Cruz-Zavala and Moreno (2019, Theorem 1), the derivative of (17) satisfies

$$\dot{V}_{\tilde{\xi}} \le -\kappa V_{\tilde{\xi}}^{\frac{p-1}{p}} \tag{18}$$

for some constant $\kappa > 0$. Separating variables and integrating inequality (18) over the time interval $0 \le \tau \le t$, we obtain

$$V_{\tilde{\xi}}^{1/p}(\tilde{\xi}(t)) \le -\frac{\kappa}{p} t + V_{\tilde{\xi}}^{1/p}(\tilde{\xi}(0)).$$

Consequently, $V_{\tilde{\xi}}$ reaches zero in a finite time $T_1 \le T_1^* := \frac{p}{\kappa} V_{\tilde{\xi}}^{1/p}(\tilde{\xi}(0))$, which implies the estimated error $\tilde{\xi}$ converges to zero in a finite time T_1. □

3.2. Reconstruction of Δ

In what follows, let us proceed with the design of an interval observer-based estimator for Δ in (8), which features the finite-time convergence property as well.

3.2.1. Interval observer design

We first design an interval observer to obtain the upper and lower boundary estimations of the state $\mathbf{e}_a$. Since it is difficult to directly compute a gain vector $M \in \mathbb{R}^r$ such that $\mathbf{A} - M\mathbf{C}$ is Hurwitz and Metzler (Raïssi, Efimov, & Zolghadri, 2012), we introduce a coordinate change $\mathbf{e}_a \mapsto \zeta := T\mathbf{e}_a$, in which the selection of the invertible matrix

$T \in \mathbb{R}^{r\times r}$ and vector M is given in Remark 3.1. Then, the dynamics of $\mathbf{e}_a$ in (7) is transformed into

$$\dot{\zeta} = T\mathbf{A}T^{-1}\zeta + T\mathbf{B}\beta u + T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d),$$
$$y = \mathbf{C}T^{-1}\zeta. \tag{19}$$

For system (19), denote $\overline{\Delta}^* = \max_{(z,\mathbf{e}_a)\in D} |\Delta(z, \mathbf{e}_a, u, d)|$, an interval observer is designed and its convergence property is given as follows:

$$\dot{\overline{\zeta}} = T(\mathbf{A} - M\mathbf{C})T^{-1}\overline{\zeta} + T\mathbf{B}\beta u + TMy + |T\mathbf{B}|\overline{\Delta},$$
$$\dot{\underline{\zeta}} = T(\mathbf{A} - M\mathbf{C})T^{-1}\underline{\zeta} + T\mathbf{B}\beta u + TMy - |T\mathbf{B}|\overline{\Delta}, \tag{20}$$

where $\overline{\zeta}, \underline{\zeta} \in \mathbb{R}^r$, and the parameter $\overline{\Delta}$ is selected such that $\overline{\Delta} \geq \overline{\Delta}^*$. The initial conditions are set as $\overline{\zeta}(0) = (TQ)^+\overline{\xi}(0) - (TQ)^-\underline{\xi}(0)$ and $\underline{\zeta}(0) = (TQ)^+\underline{\xi}(0) - (TQ)^-\overline{\xi}(0)$ in which $Q \in \mathbb{R}^{r\times r}$ is an invertible transformation matrix such that $\xi \mapsto \mathbf{e}_a = Q\xi$ in (4).

Lemma 3.2. *Suppose that* $(z, \mathbf{e}_a) \in \mathcal{D}$ *for all* $t \geq 0$ *and Assumption 2.3 holds. If the transformation matrix* T *and the gain vector* M *are selected such that* $T(\mathbf{A} - M\mathbf{C})T^{-1}$ *is not only Hurwitz but also Metzler, then the dynamics of (20) satisfies* $\underline{\zeta} \leq \zeta \leq \overline{\zeta}$ *for all* $t \geq 0$*. Moreover,* $\mathbf{e}_a$ *in (7) satisfies* $\underline{\mathbf{e}}_a \leq \mathbf{e}_a \leq \overline{\mathbf{e}}_a$ *for all* $t \geq 0$ *with the coordinate change as* $\overline{\mathbf{e}}_a := (T^{-1})^+\overline{\zeta} - (T^{-1})^-\underline{\zeta}$ *and* $\underline{\mathbf{e}}_a := (T^{-1})^+\underline{\zeta} - (T^{-1})^-\overline{\zeta}$.

The proof can be found in Appendix.

Remark 3.1. The construction of an invertible matrix T and a vector M such that $T(\mathbf{A} - M\mathbf{C})T^{-1}$ is not only Hurwitz but also Metzler can be followed by the procedures in Efimov, Perruquetti, Raïssi, and Zolghadri (2013). It can be concluded as follows:

(i) Find a Hurwitz and Metzler matrix $\Lambda \in \mathbb{R}^{r\times r}$ arbitrarily provided that Λ and $\mathbf{A}$ have no common eigenvalues. Then, set $\Lambda = T(\mathbf{A} - M\mathbf{C})T^{-1}$, that is equivalent to $T\mathbf{A} - \Lambda T = TM\mathbf{C}$.
(ii) Choose an arbitrary vector $S \in \mathbb{R}^r$, and solve the Sylvester equation $T\mathbf{A} - \Lambda T = S\mathbf{C}$ to obtain T.
(iii) Finally, utilize the relation $S = TM$ and calculate $M = T^{-1}S$. $\triangleleft$

3.2.2. Algebraic expression of Δ

Next, reposing on the preceding estimator $\hat{e}_r$ in (14) and the interval observer (20), we are ready to employ an algebraic unknown input reconstruction method proposed in Zhu et al. (2023) to obtain the finite-time estimator $\hat{\Delta}$.

From Lemma 3.2, we have $\underline{\mathbf{e}}_a \leq \mathbf{e}_a \leq \overline{\mathbf{e}}_a$, which implies $\underline{e}_r \leq e_r \leq \overline{e}_r$ with $\underline{e}_r, \overline{e}_r$ being the rth element of $\underline{\mathbf{e}}_a$ and $\overline{\mathbf{e}}_a$ respectively. Hence, one concludes there exists a time-varying scalar $\rho(t)$ satisfying $0 \leq \rho \leq 1$ such that the equation $e_r = \rho\overline{e}_r + (1-\rho)\underline{e}_r$ holds for all $t \geq 0$. We rewrite such an equation into

$$e_r = \rho(\overline{e}_r - \underline{e}_r) + \underline{e}_r = \rho f_1(\tilde{\zeta}) + \underline{e}_r, \tag{21}$$

in which $f_1(\tilde{\zeta}) := \mathbf{B}^\top|T^{-1}|\tilde{\zeta}$ and $\tilde{\zeta} := \overline{\zeta} - \underline{\zeta}$. The second equation of (21) follows from the fact that $\overline{e}_r - \underline{e}_r = \mathbf{B}^\top(\overline{\mathbf{e}}_a - \underline{\mathbf{e}}_a)$ and $(T^{-1})^+ + (T^{-1})^- = |T^{-1}|$.

Then, differentiating (21) gives

$$\dot{e}_r = \rho f_2(\tilde{\zeta}) + \dot{\rho} f_1(\tilde{\zeta}) + \dot{\underline{e}}_r, \tag{22}$$

in which $f_2(\tilde{\zeta})$ can be obtained by subtracting the dynamics of $\underline{\zeta}$ from that of $\overline{\zeta}$ in (20), that is

$$f_2(\tilde{\zeta}) = \dot{f}_1(\tilde{\zeta}) = \mathbf{B}^\top|T^{-1}|\left(T(\mathbf{A} - M\mathbf{C})T^{-1}\tilde{\zeta} + 2|T\mathbf{B}|\overline{\Delta}\right).$$

Using $\underline{e}_r = \mathbf{B}^\top\underline{\mathbf{e}}_a$ and with the dynamics of $\overline{\zeta}$ and $\underline{\zeta}$ in (20), the dynamics of $\underline{e}_r$ is

$$\dot{\underline{e}}_r = f_3(\overline{\zeta}, \underline{\zeta}, y) + \beta u \tag{23}$$

with

$$f_3(\overline{\zeta}, \underline{\zeta}, y) := \mathbf{B}^\top\left(N_1\underline{\zeta} - N_2\overline{\zeta} + My - |T^{-1}||T\mathbf{B}|\overline{\Delta}\right),$$

$$N_1 := (T^{-1})^+T(\mathbf{A} - M\mathbf{C})T^{-1},$$
$$N_2 := (T^{-1})^-T(\mathbf{A} - M\mathbf{C})T^{-1}.$$

Meanwhile, reminiscent of the dynamics of e_r in (7), we have $\dot{e}_r = \beta u + \Delta(z, \mathbf{e}_a, u, d)$, that together with (22), gives the algebraic expression of $\Delta(z, \mathbf{e}_a, u, d)$ in the form of

$$\Delta(z, \mathbf{e}_a, u, d) = \rho f_2(\tilde{\zeta}) + \dot{\rho} f_1(\tilde{\zeta}) + f_3(\overline{\zeta}, \underline{\zeta}, y). \tag{24}$$

However, we are still not able to employ (24) for implementation since the signals $\rho, \dot{\rho}$ are unavailable for measurement. Instead, according to (24), we proposed the estimator for $\Delta(z, \mathbf{e}_a, u, d)$ in (8) as

$$\hat{\Delta} = \hat{\rho} f_2(\tilde{\zeta}) + \hat{\dot{\rho}} f_1(\tilde{\zeta}) + f_3(\overline{\zeta}, \underline{\zeta}, y) \tag{25}$$

in which $\hat{\rho}$ and $\hat{\dot{\rho}}$ are estimates of ρ and $\dot{\rho}$ respectively, functions $f_i(\cdot), i = 1, 2, 3$ are found in (21)–(23) with $\overline{\zeta}, \underline{\zeta}$ given in (20).

To proceed, based on (21), we can deduce that

$$\hat{\rho} = \frac{\hat{e}_r - \underline{e}_r + \epsilon}{\overline{e}_r - \underline{e}_r + \epsilon} \tag{26}$$

with $\epsilon = 1$, if $\overline{e}_r = \underline{e}_r$, and otherwise, $\epsilon = 0$. The design of the estimator $\hat{\dot{\rho}}$ again relies on a HOSM differentiator in the form of

$$\dot{\psi}_1 = \psi_2 - k_1' L'^{\frac{1}{2}} \lceil \psi_1 - \hat{\rho} \rfloor,$$
$$\dot{\psi}_2 = -k_2' L' \lceil \psi_1 - \hat{\rho} \rfloor^0$$
$$\hat{\dot{\rho}} = \psi_2 \tag{27}$$

where k_1', k_2' are positive parameters, and $L' > 0$ is chosen to be sufficiently large. The convergence property of estimated error $\tilde{\Delta} = \hat{\Delta} - \Delta$ is asserted by the following proposition.

Proposition 3.1. *Given* $(z, \mathbf{e}_a) \in \mathcal{D}$ *for all* $t \geq 0$ *and let Assumption 2.3 hold. Consider the estimator* $\hat{\Delta}$ *in (25) that is composed of the HOSM differentiators in (15), (27) and interval observer in (20), there exist a positive constant* L'^* *and a time instant* $T_2 > T_1$ *such that for all* $L' \geq L'^*$ *and parameters* k_1', k_2' *selected properly, the estimated error* $\tilde{\Delta}$ *converges to zero in a finite time* T_2.

Proof. From (24) and (25), it follows that

$$\tilde{\Delta} = \tilde{\rho} f_2(\tilde{\zeta}) + \tilde{\dot{\rho}} f_1(\tilde{\zeta}) \tag{28}$$

in which $\tilde{\rho} := \hat{\rho} - \rho = \frac{\tilde{\xi}_r + \sum_{i=1}^{r-1} k^{r-i}\alpha_{i-1}\tilde{\xi}_i}{\overline{e}_r - \underline{e}_r + \epsilon}$ and $\tilde{\dot{\rho}} := \hat{\dot{\rho}} - \dot{\rho}$. The former dynamics is governed by (16) that converges to zero in a finite time T_1 while the latter one is given by

$$\dot{\tilde{\psi}}_1 = \tilde{\psi}_2 - k_1' L'^{\frac{1}{2}} \lceil \tilde{\psi}_1 - \tilde{\rho} \rfloor,$$
$$\dot{\tilde{\psi}}_2 = -k_2' L' \lceil \tilde{\psi}_1 - \tilde{\rho} \rfloor^0 - \ddot{\rho}$$
$$\tilde{\dot{\rho}} = \tilde{\psi}_2 \tag{29}$$

in which $\tilde{\psi} := (\tilde{\psi}_1, \tilde{\psi}_2)$ with $\tilde{\psi}_1 := \psi_1 - \rho$ and $\tilde{\psi}_2 := \psi_2 - \dot{\rho}$. By seeking second derivatives on both sides of (21), one deduces that the boundedness of signal $\ddot{\rho}$ relies on the boundedness of the ith derivatives of signals $e_r, \underline{e}_r, \overline{e}_r$, $i = 0, 1, 2$. Since $(z, \mathbf{e}_a) \in \mathcal{D}$, $b(z, \xi)$ is bounded from (3), d is bounded under Assumption 2.3, $\hat{\xi}$ is bounded from Lemma 3.1 and the bounded state ξ, and thanks to boundedness of Sat$[\cdot]$ in (13), signals $e_r, \underline{e}_r, \overline{e}_r$ and their first derivatives are bounded. Besides, since $\dot{d}$ is bounded under Assumption 2.3, $\dot{\hat{\Delta}}$ is bounded and by the smoothness of Sat$[\cdot]$ in (13), the second derivatives of $e_r, \underline{e}_r, \overline{e}_r$ are bounded as well, which implies the boundedness of signal $\ddot{\rho}$. Again, according to Levant (2003), if the gain L' is selected such that $L' \geq \max_{(z,\mathbf{e}_a)\in D} \ddot{\rho}$, one can choose the gains $k_2' = 1.1$, $k_1' = 1.5$. Then, similar to the Lyapunov candidate function constructed in (17), we consider

$$V_{\tilde{\psi}} = \beta_1' Z_1'(\tilde{\psi}_1, \tilde{\psi}_2) + \beta_2' \frac{1}{p'} |\tilde{\psi}_2|^{p'} \tag{30}$$

in which $\beta_1', \beta_2' > 0$, $p' > 1$, $Z_1'(\tilde{\psi}_1, \tilde{\psi}_2) = \frac{2}{p'}|\tilde{\psi}_1|^{\frac{p'}{2}} - \tilde{\psi}_1\lceil\tilde{\psi}_2\rfloor^{\frac{p'-2}{3}} + (\frac{p'-2}{p'})|\tilde{\psi}_2|^{\frac{p'}{3}}$. Based on Cruz-Zavala and Moreno (2019, Proposition 2), $\tilde{\psi}$ satisfies the following inequality

$$\|\tilde{\psi}_i(t)\| \le \lambda_i L'^{\frac{i-1}{2}} (\max_{t\ge 0}\|\tilde{\rho}\|)^{\frac{3-i}{2}}, \quad i = 1, 2$$

with $\lambda_i \ge 1$ depending only on the gains k_i', which implies the boundedness of $\tilde{\psi}$ and ψ_1, ψ_2. What is more, after the finite time T_1 defined in Lemma 3.1, $\tilde{\rho} = 0$. Then, by Cruz-Zavala and Moreno (2019, Theorem 1), the derivative of (30) satisfies

$$\dot{V}_{\tilde{\psi}} \le -\kappa' V_{\tilde{\psi}}^{\frac{p'-1}{p'}} \tag{31}$$

for some $\kappa' > 0$ after $t \ge T_1$. From (31), one can easily conclude that after another period, say $t_f := \frac{p'}{\kappa'} V_{\tilde{\psi}}^{1/p'}(\tilde{\psi}(T_1))$, $\tilde{\psi} = 0$. Then, setting $T_2 = T + t_f$, from (28) we obtain $\tilde{\Delta} = 0$ for all $t \ge T_2$. □

Remark 3.2. Most of UIO techniques (Alenezi et al., 2021; Corless & Tu, 1998; Yang et al., 2022) are developed mainly for the estimation problem such as fault diagnosis, which requires the exact model of the plant. We consider the class of systems in the form of (1), which satisfies the minimum phase condition and the observer matching condition required for an UIO design problem. However, instead of directly estimating the state and disturbances, we regard the model uncertainties and disturbances as the lumped uncertainty to be compensated, which makes our method robust to the model uncertainties. ◁

Remark 3.3. Consider a stabilization problem, the interval observer-based UIO provides the controller with a control-decoupled estimation, which makes it possible to introduce the unknown input reconstruction into the controller. On the other hand, the combination of the HOSM differentiators (14), (27) renders the UIO the finite-time convergence. ◁

4. Stability analysis

The previous section has provided constructive procedures of the estimated term $\hat{e}_r$ and $\hat{\Delta}$ required in (12). Their convergence properties are concluded in Lemma 3.1 and Proposition 3.1 under the temporary assumption that $(z, \mathbf{e}_a) \in \mathcal{D}$. Now, the closed-loop system (7) under the control protocol (12) is in the form of

$$\begin{aligned} &\dot{z} = f_0(z, e_1), \qquad \dot{\mathbf{e}} = kA\mathbf{e} + Be_r, \\ &\dot{e}_r = -\frac{b(z,\xi)}{\beta}(l\hat{e}_r + \mathrm{Sat}[\hat{\Delta}]) + \delta(z, \mathbf{e}, e_r, d). \end{aligned} \tag{32}$$

In this section, we will mathematically demonstrate the existence of such a forward invariant set $\mathcal{D}$ under the condition that l in (12) is chosen to be sufficiently large.

Before stepping into the stability analysis of the closed-loop system, let us turn our attention to the augmented zero dynamics

$$\dot{z}_a = \mathbf{f}_0(z_a) \tag{33}$$

with $z_a := (z, \mathbf{e})$ and $\mathbf{f}_0(z_a) := (f_0(z, e_1), kA\mathbf{e})$. Under Assumption 2.1, the results in Byrnes and Isidori (1991), Isidori et al. (2012) show that the origin of system (33) is uniformly semi-globally locally asymptotically stable in the parameter k. For future use, let $\mathcal{B}_r$ and $\overline{\mathcal{B}}_r$ denote respectively the open and the closed ball of radius r around the origin of system (33).

Lemma 4.1 (*Byrnes & Isidori, 1991, Theorem 7.3*). *Under Assumption 2.1, for each positive number $0 < R < \infty$, there exist positive constants a, r, k^*, and, for any $k \ge k^*$, a continuously differentiable, positive definite function $W(z_a)$ such that*

(i) the set $\{z_a \in \mathbb{R}^{n+r-1} : W(z_a) \le a\}$ is compact, and contains $\overline{\mathcal{B}}_r$ in its interior;

(ii) the derivative of $W(z_a)$ along the trajectories of (33) satisfies

$$\dot{W}(z_a) \le -\theta(W(z_a)), \quad \forall z_a \in \mathcal{B}_r \tag{34}$$

for some class $\mathcal{K}_\infty$ function $\theta(\cdot)$ that is dependent on the positive parameter k.

Following Lemma 4.1, we present the main result:

Theorem 4.1. *Suppose Assumptions 2.1–2.3 hold. Given $\overline{\mathrm{Z}}_0$ and $\overline{\mathrm{X}}_0$ being the compact set, for any uniformly bounded disturbance d, there exist positive constants k^*, l^*, L^*, L'^*, ϑ and a compact set $\mathcal{D}$ such that for all $0 < \beta \le \underline{b}$, $k \ge k^*$, $l \ge l^*$, $L \ge L^*$, $L' \ge L'^*$, the trajectories of the closed-loop system (32) originating from $\overline{\mathrm{Z}}_0 \times \overline{\mathrm{X}}_0$ are bounded and the regulated output exponentially converges to zero.*

Proof. The main idea of the proof relies on the following observation of dynamics (32): given a bounded e_r, if k is chosen to be sufficiently large, one can constrain state z_a within certain compact set. On the other hand, one can easily see the magnitude of e_r is adjustable by sufficiently large l given δ is a bounded term. In summary, we will show that the tuning gains k and l suffice to ensure the existence of $\mathcal{D}$. Next is to prove the control input u in (12) becomes unsaturated after the estimated errors converge to zero, then the closed-loop system is recovered to the stabilized system (11).

First, we define the forward invariant set $\mathcal{D}$ as $\mathcal{D} := W^{-1}([0, c+1]) \times \mathcal{B}_{c+1}$, in which $W : \{z_a \in \mathbb{R}^{n+r-1} : W(z_a) \le a\} \to \mathbb{R}$ is the Lyapunov function given by Lemma 4.1 and with $\mathbf{b} := |\frac{b(z,\xi)-\beta}{b(z,\xi)}|$,

$$c = \frac{1}{1-\mathbf{b}}(|e_r(0)| + \max_{t\ge 0}|\tilde{e}_r|)$$

Since

$$0 < \beta \le \underline{b}, \tag{35}$$

one obtains $0 \le \mathbf{b} < 1$, and thus $c > 0$ is a valid radius such that $\overline{\mathrm{X}}_0 \subset \mathcal{B}_c := \{e_r \in \mathbb{R} : |e_r| \le c\}$ and $\overline{\mathrm{Z}}_0 \subset W^{-1}([0, c])$. In what follows, we will prove the trajectories of (32) originating from $\overline{\mathrm{Z}}_0 \times \overline{\mathrm{X}}_0 \subset W^{-1}([0, c]) \times \mathcal{B}_c$ will be restricted to the closed set $\mathcal{D} := W^{-1}([0, c+1]) \times \mathcal{B}_{c+1}$.

Recalling the form of B in (5) and when $r > 1$, the derivative of $W(z_a)$ along the trajectory of (32) satisfies

$$\begin{aligned} \dot{W}(z_a) &\le -\theta(W(z_a)) + m_1 k^{-(r-2)} \max_{t\ge 0}|e_r| \\ &\le -(1-\pi)\theta(W(z_a)) - \pi\theta(W(z_a)) + m_1 k^{-(r-2)} \max_{t\ge 0}|e_r| \end{aligned} \tag{36}$$

where $0 < \pi < 1$ and $m_1 := \max_{z_a \in W^{-1}([0,c+1])} |\frac{\partial W(z_a)}{\partial z_a}|$ independent of k. Since $\theta(\cdot)$ is dependent on k, there exists a k^* such that for $k \ge k^* > 0$, one obtains $m_2 \le c+1$, in which $m_2 := \theta^{-1}(mk^{-(k-2)} \max_{t\ge 0}|e_r|/\pi)$. Hence, reverting back to (36), we derive

$$\begin{aligned} &\dot{W}(z_a) \le -(1-\pi)\theta(W(z_a)), \\ &\forall z_a \in \{z_a \in \mathbb{R}^{n+r-1} : m_2 \le W(z_a) \le c+1\}, \end{aligned} \tag{37}$$

which implies z_a is constrained within the compact set $W^{-1}([0, c+1])$. When $r = 1$, this can be achieved if the input e_r is sufficiently small.

Next, we show that with a sufficiently large l, e_r remains in $\mathcal{B}_{c+1}$. To this end, set

$$\vartheta = \max_{(z_a, e_r)\in\mathcal{D}} \mathbf{b}l|e_r| + \frac{\beta}{b(z,\xi)}|\delta(z, \mathbf{e}, e_r, d)| \tag{38}$$

with $\delta(\cdot)$ defined in (6). Now, consider the dynamics of e_r in (32) and replace $\mathrm{Sat}[\hat{\Delta}]$ with (38), it follows that

$$\begin{aligned} |e_r(t)| &\le e^{-\frac{b(\cdot)}{\beta}lt}|e_r(0)| \\ &+ (1 - e^{-\frac{b(\cdot)}{\beta}lt})\left[\max_{t\ge 0}|\tilde{e}_r| + \max_{(z_a,e_r)\in\mathcal{D}} \mathbf{b}|e_r| + \frac{2\beta}{b(\cdot)l}|\delta(\cdot)|\right] \end{aligned} \tag{39}$$

where we omit the augments in $b(\cdot)$, $\delta(\cdot)$ and $\Delta(\cdot)$ for simplicity. Taking the maximum values of both sides of the inequality (39), it holds:

$$\max_{(z_a,e_r)\in\mathcal{D}} |e_r| \le |e_r(0)| + \max_{t\ge 0}|\tilde{e}_r| + \max_{(z_a,e_r)\in\mathcal{D}} \mathbf{b}|e_r| + \frac{2\beta}{b(\cdot)l}|\delta(\cdot)|$$

which yields

$$\max_{(z_a,e_r)\in D}|e_r| \leq c + \frac{2\beta}{b(\cdot)(1-\mathbf{b})l}\max_{(z_a,e_r)\in D}|\delta(\cdot)|.$$

Hence, select

$$l \geq l^* = \frac{2\beta\max_{(z_a,e_r)\in D}|\delta(\cdot)|}{b(\cdot)(1-\mathbf{b})} \tag{40}$$

such that $\frac{2\beta}{b(\cdot)(1-\mathbf{b})l}\max_{(z_a,e_r)\in D}|\delta(\cdot)| \leq 1$, which implies that

$$|e_r(t)| \leq \max_{(z_a,e_r)\in D}|e_r| \leq c+1. \tag{41}$$

Thus, with a sufficiently large l, the input of the augmented zero dynamics, e_r, can be sufficiently small. From (37) and (41), it turns out that, with a consequence of the choice of β, k, l, ϑ, the trajectories of (32) originating from $\overline{Z}_0 \times \overline{X}_0$ remain in $\mathcal{D}$.

The last thing is to show the convergence property of the regulated output. Since we have proven $(z, \mathbf{e}_a) \in \mathcal{D}, \forall t \geq 0$, it is shown in Lemmas 3.1 and 3.1 that there exist L^*, L'^* and a finite time instant T such that for all $L \geq L^*$, $L' \geq L'^*$, one derives $\tilde{e}_r = 0$ and $\tilde{\Delta} = 0$, $\forall t \geq T$. Bearing in mind the definitions of $\Delta(\cdot)$ and u in (12) and (24), we obtain $\Delta(\cdot) = -(b(\cdot)-\beta)\frac{l}{b(\cdot)}e_r + \frac{\beta}{b(\cdot)}\delta(\cdot)$ after $t \geq T$. Thanks to the saturation value set in (38), it implies that $\mathrm{Sat}[\Delta(\cdot)] = \Delta(\cdot)$ after $t \geq T$ and the closed-loop system (32) becomes (11). Therefore, the targeted closed-loop system is asymptotically stable. Besides, the regulated output $y = e_1 = \xi_1$ exponentially converges to zero, which ends the proof. $\square$

Remark 4.1. The expression for the radius of the forward invariant set is given by $c = \frac{1}{1-\mathbf{b}}(|e_r(0)| + \max_{t\geq 0}|\tilde{e}_r|)$, where the radius depends on the extent of uncertainties, including initial conditions of the state and estimated error. Moreover, in the case of large uncertainty in the nonlinear function $\delta(\cdot)$, it becomes essential to design a larger value for l such that $\frac{2\beta}{b(\cdot)(1-\mathbf{b})l}\max_{(z_a,e_r)\in D}|\delta(\cdot)| \leq 1$. Consequently, the selection of l depends on the amount and quality of the available information on the system and disturbance. In turn, if uncertainties are substantial, a higher gain l should be chosen accordingly, and vice versa. $\triangleleft$

5. Numerical example

In this section, we first conduct a numerical experiment to validate the proposed method. Then, in order to demonstrate the effectiveness of the proposed method, comparative studies with the recently proposed adaptive IM-based method in Bernard et al. (2020) and a nonlinear DOB-based controller in Back and Shim (2008) are carried out. All the simulations are conducted under ode45 with the identical simulation precision of 1×10^{-3}.

5.1. Case I : Exact rejection of unknown disturbance

The system is in the form of

$$\dot{z} = -2z + y + 2\phi w_1,$$
$$\dot{y} = w_2^2 + zy + u$$

with y to be regulated to zero, and a disturbance $d = w_2^2$ generated by an exosystem of

$$\dot{w}_1 = w_2,$$
$$\dot{w}_2 = (1 - w_1^2)w_2 - w_1.$$

We consider $\phi = 0$ in the first case. Assume that the initial condition of regulated output y ranges over a known compact set $\{y \in \mathbb{R} : 0 \leq y \leq 12\}$ and the upper boundaries of d and $\dot{d}$ are known, here $|d(t)| \leq 8$ and $|\dot{d}(t)| \leq 18$, which is depicted in Fig. 2. Under the above settings, a streamlined procedure to construct the proposed UIO-based controller is as follows.

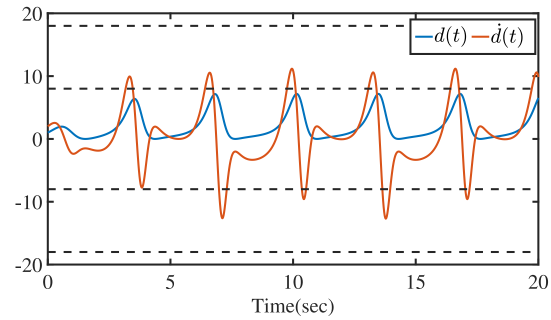

Fig. 2. Time history of the disturbance $d(t)$ and its derivative $\dot{d}(t)$.

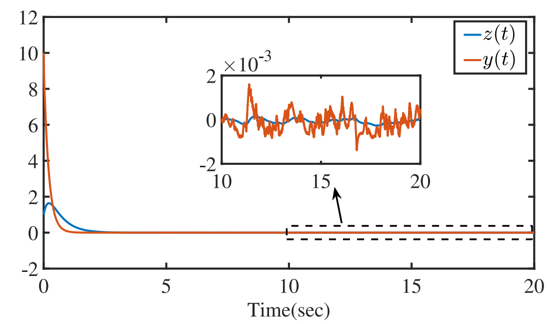

Fig. 3. Time history of regulated output $y(t)$.

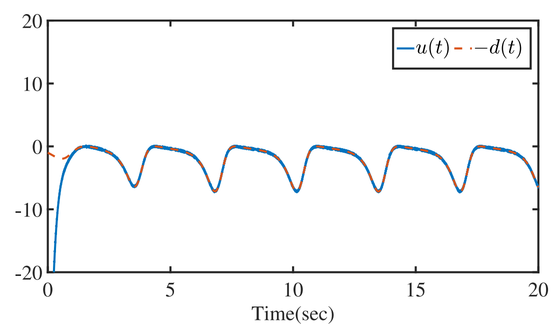

Fig. 4. Time history of disturbance $d(t)$ and control input $u(t)$.

First, set $\beta = 0.5$ such that $\mathbf{b} = 0.5$. A sufficiently large parameter l is chosen to be $l = 5$. The compact set $\mathcal{D}$ is set as $\mathcal{D} = \{(z, y) \in \mathbb{R}^2 : |z| \leq 2, |y| \leq 12\}$. Hence, using the fact that

$$\delta(z, y, d) = d + zy,$$

from (38) it can be verified that

$$\max_{(z,y)\in D} \mathbf{b}l|y| + \beta|d + zy| \leq 46.$$

Here, we take the saturation value as $\vartheta = 50$. The controller (12) is in the form of

$$u = -10y - 2\,\mathrm{Sat}[\hat{\Delta}]$$

where the estimate $\hat{\Delta}$ is provided by (25), which is composed of the HOSM differentiator (27) and the interval observer (20). As for the former, a sufficiently large L' is chosen to be $L' = 20$ and set the turning gains $k_1' = 1.5, k_2' = 1.1$. As for the latter, the gain vector M and the transformation matrix T are selected according to Remark 3.1 as $M = 4$ and $T = 0.25$ such that $T(\mathbf{A} - M\mathbf{C})T^{-1} = -4$ is Hurwitz and Metzler.

After the above parameters are fixed, we conduct the simulation with initial conditions $(z(0), y(0)) = (1, 10)$, $(w_1(0), w_2(0)) = (0, 1)$, $(\psi_1(0), \psi_2(0)) = (0.5, 0)$, and $(\overline{\zeta}(0), \underline{\zeta}(0)) = (20, 0)$.

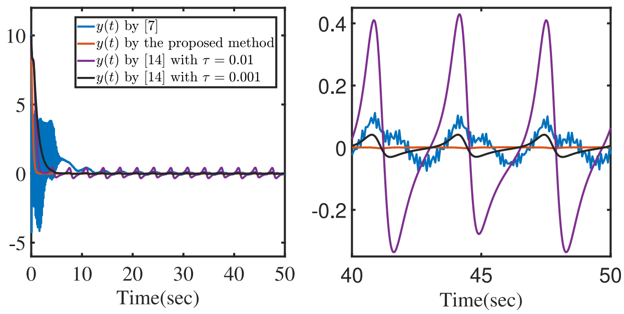

Fig. 5. Time history of regulated output $y(t)$ with the adaptive IM-based method (Bernard et al., 2020), the proposed method and the DOB-based method in Back and Shim (2008).

Figs. 3–4 show the results of Case I. Remarkably, the regulated output y converges to zero with a notable transient performance and the control input u completely reconstructs the disturbance d in a rather short time, which shows the effectiveness of the proposed method.

5.2. Case II : Comparison with Other Disturbance Rejection Methods Bernard et al. (2020) and Back and Shim (2008)

In the second case, we set $\phi = 1$, thus the same system as Bernard et al. (2020) is considered here for the sake of fair comparison. With the same exosystem, all the tuning parameters and initial conditions remain the same, except for the initial condition of the exosystem set as $(w_1(0), w_2(0)) = (0, 4)$.

For compared methods, under the same preconditions, the gains in the IM-based method (Bernard et al., 2020) are set as $\Omega = 10^{-6}\mathrm{I}$, $\mathbf{p} = 0.05$, $k = 5$, $L = 15$ with the saturation value set at 250. Finally, the tuning parameters in the nonlinear DOB-based controller (Back & Shim, 2008) are chosen as $\rho = 0.1$ and $\tau = 0.01$ with the saturation value taken at 40.

The results are shown in Fig. 5. For the adaptive IM-based method (Bernard et al., 2020), the residual error exists in the regulated output due to the ability of the identifier. Specifically, when the exosystem cannot be exactly identified by the selected identifier, only approximate regulation is achieved. Furthermore, the selection of the least-square identifier with a low forgetting factor $\mathbf{p}$ yields a longer convergence time. On the other hand, the steady-state error of the DOB-based method (Back & Shim, 2008) highly relies on the choice of the cut-off frequency τ. Distinct from these two approaches, our proposed method features the exact compensation of the disturbance and does not require tending some parameters to infinity or zero. Additionally, if the relative degree of the system is high, the observer gain in Bernard et al. (2020) and Back and Shim (2008) grows rapidly, which poses a challenge in the numerical implementation. Despite the residual error in the output shown in Figs. 3 and 5, it can be significantly improved with the higher simulation precision. In summary, under the proposed controller, the controller system maintains robust stability, and the regulation performance is significantly improved compared with that of the design in Bernard et al. (2020) and in Back and Shim (2008).

6. Conclusions

In this paper, with a novel unknown input observer-based controller, exponential regulation of the regulated output for uncertain minimum phase systems has been achieved. Through a delicately designed coordinate change, we cast the output regulation problem as the problem of estimating the unknown input. Different from the existing robust control techniques that only achieve the approximate regulation, the exact unknown input observer-based controller enables the original system to recover a disturbance-free asymptotically stable system. Then, the regulated output exponentially converges to zero in a semi-global setting, which is theoretically proven and numerically verified. Moreover, with comparisons to both internal model-based method and disturbance observer-based approach in a numerical experiment, the proposed method has the advantage of a satisfactory convergence rate and robustness to the model uncertainties and external disturbances. In future work, the extension to multivariable systems and the removal of the minimum phase requirement will be our research directions.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Proof of Lemma 3.2

To this end, two necessary lemmas are introduced as follows:

Lemma A.1 (Farina & Rinaldi, 2011). *Suppose that the matrix $A \in \mathbb{R}^{n\times n}$ is a Metzler and Hurwitz matrix, besides $d_x(t) \in \mathbb{R}^n$, $d_x(t) \ge 0$, $t \ge 0$, then the solution of the dynamics $\dot{x}(t) = Ax(t) + d_x(t)$ satisfies $x(t) \ge 0$ for all $t \ge 0$ if $x(0) \ge 0$.*

Lemma A.2 (Mazenc, Dinh, & Niculescu, 2014). *Suppose that the vector variables $\underline{x}(t) \in \mathbb{R}^n$, $x(t) \in \mathbb{R}^n$ and $\overline{x}(t) \in \mathbb{R}^n$ satisfies $\underline{x}(t) \le x(t) \le \overline{x}(t)$, then for any constant matrix $M \in \mathbb{R}^{m\times n}$, we have $M^+\underline{x}(t) - M^-\overline{x}(t) \le Mx(t) \le M^+\overline{x}(t) - M^-\underline{x}(t)$.*

Define $\overline{\zeta}_e := \overline{\zeta} - \zeta$ and $\underline{\zeta}_e := \zeta - \underline{\zeta}$. From (19) and (20), it follows that

$$\dot{\overline{\zeta}}_e = T(\mathbf{A} - M\mathbf{C})T^{-1}\overline{\zeta}_e + |T\mathbf{B}|\overline{\Delta} - T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d),$$
$$\dot{\underline{\zeta}}_e = T(\mathbf{A} - M\mathbf{C})T^{-1}\underline{\zeta}_e + T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d) + |T\mathbf{B}|\overline{\Delta}.$$

By definition of $|T\mathbf{B}| = (T\mathbf{B})^+ + (T\mathbf{B})^-$, we have

$$|T\mathbf{B}|\overline{\Delta} = (T\mathbf{B})^+\overline{\Delta} - (T\mathbf{B})^-(-\overline{\Delta}) = -(T\mathbf{B})^+(-\overline{\Delta}) + (T\mathbf{B})^-\overline{\Delta}.$$

Hence, by Lemma A.2, we obtain

$$|T\mathbf{B}|\overline{\Delta} - T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d) = (T\mathbf{B})^+\overline{\Delta} - (T\mathbf{B})^-(-\overline{\Delta}) - T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d) \ge 0.$$

Similarly, $T\mathbf{B}\Delta(z, \mathbf{e}_a, u, d) + |T\mathbf{B}|\overline{\Delta} \ge 0$. Under Assumption 2.3 that $\underline{\xi}(0) \le \xi(0) \le \overline{\xi}(0)$, we again utilize Lemma A.2 and $\zeta(0) = TQ\xi(0)$, thus $\underline{\zeta}(0) \le \zeta(0) \le \overline{\zeta}(0)$. Therefore, based on Lemma A.1, from the dynamics of $\underline{\zeta}_e$, $\overline{\zeta}_e$, if the matrix $T(\mathbf{A} - M\mathbf{C})T^{-1}$ is a Metzler and Hurwitz matrix, we can conclude that $\underline{\zeta}_e \ge 0$ and $\overline{\zeta}_e \ge 0$ for all $t \ge 0$. Thus, $\underline{\zeta} \le \zeta \le \overline{\zeta}$ for all $t \ge 0$. Finally, employing the relation $\mathbf{e}_a = T^{-1}\zeta$ and Lemma A.2, it follows that $\underline{\mathbf{e}}_a \le \mathbf{e}_a \le \overline{\mathbf{e}}_a$ for all $t \ge 0$ with $\underline{\mathbf{e}}_a$, $\overline{\mathbf{e}}_a$ defined in Lemma 3.2. □

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